$$\int_0^1 g(x)\,dx = g(\xi).$$

Exercise A.32 *Adjust the procedure from Example A.31 to prove the equivalence of the maximum norm (A.17) and the p-norm (A.19) with $p = 2$ in the space $P^k([-1, 1])$, where k is an arbitrary natural number.*

Exercise A.33 *Let $(a, b) \subset \mathbb{R}$ be a nonempty bounded interval.*

1. *Construct an infinite sequence of functions in the space $V = C(a, b)$ that converges in the p-norm for all $1 \le p < \infty$ but which does not converge in the maximum norm.*
2. *Use this sequence and Definition A.34 to show that the maximum and p-norms are not equivalent in V.*
3. *Is it possible to construct a sequence in V which converges in the maximum norm but does not converge in the p-norm? Present a proof.*
4. *Find a subset $S \subset V$ that is open in the maximum norm but is not open in the p-norm. Is it possible to do this vice versa as well?*

Exercise A.34 *Show that the definitions of the operator norm (A.26) and (A.27) are equivalent.*

Exercise A.35 *Prove Proposition A.6 (every convergent sequence in a normed space is a Cauchy sequence).*

Exercise A.36 *Let V be a normed space and $\{u_n\}_{n=1}^{\infty} \subset V$ a Cauchy sequence. Suppose that there is a subsequence $\{u_{n_k}\}_{k=1}^{\infty} \subset \{u_n\}_{n=1}^{\infty}$ and some element $u \in V$ such that*

$$\lim_{k\to\infty} u_{n_k} = u.$$

Show that

$$\lim_{n\to\infty} u_n = u.$$

Exercise A.37 *Show that the sequence (A.34) is convergent and that the limit is $\sqrt{a}$. Hint: Use, for example, (A.35).*

Exercise A.38 *Consider the open ball $B(\mathbf{0}, R) \subset \mathbb{R}^d$, $d = 3$, with a finite radius $R > 0$, and the function $f(\boldsymbol{x}) = 1/r^{\alpha}$, $r(\boldsymbol{x}) = \sqrt{x_1^2 + \ldots + x_d^2}$. Show that the following statements hold:*

1. *Let $\Omega = B(\mathbf{0}, R)$. Then $f(\boldsymbol{x}) \in L^p(\Omega)$ if and only if $\alpha < d/p$.*
2. *Let $\Omega = \mathbb{R}^d \setminus \overline{B(\mathbf{0}, R)}$. Then $f(\boldsymbol{x}) \in L^p(\Omega)$ if and only if $\alpha > d/p$.*

Describe in detail the application of the Substitution Theorem for integration in spherical coordinates and calculate the corresponding finite integrals.