■ EXAMPLE A.22 (Convergence and limit)

1. Consider the space $V = \mathbb{R}^3$ equipped with the discrete maximum norm (A.10) and a sequence of vectors $\{u_n\}_{n=1}^{\infty} \subset V$,

$$u_n = \left(1 - \frac{1}{n}, e^{-n}, \frac{\sin(n)}{n^2}\right)^T.$$

The only candidate for a limit is $v = (1, 0, 0)^T$. The sequence converges to v since

$$\|u_n - v\|_\infty \le \frac{1}{n} \quad \text{for all } n \ge 1.$$

2. Let $V = C([0, 1])$ equipped with the maximum norm (A.17),

$$\|v\|_\infty = \sup_{x \in (0,1)} |v(x)|,$$

and consider a sequence of functions $\{u_n\}_{n=1}^{\infty} \subset V$,

$$u_n(x) = x^n(1 - x) + x^3 + 1.$$

Since

$$\max_{x \in (0,1)} |u_n - (x^3 + 1)| = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \le \frac{1}{n+1},$$

the only candidate for a limit is $v(x) = x^3 + 1$. Since

$$\|u_n - v\|_\infty \le \frac{1}{n+1},$$

the sequence converges to v.

■ EXAMPLE A.23 (Nonconvergent sequences)

It is easy to show, using Lemma A.20, that the following sequences do not converge.

1. Consider the space $V = P^2([0, 1])$ equipped with the maximum norm (A.17), and the sequence $\{u_n\}_{n=1}^{\infty} \subset V$, $u_n(x) = nx(1 - x)$. The sequence is not bounded ($\|u_n\|_\infty = n/4$ for all n).

2. In the same space V let $\{u_n\}_{n=1}^{\infty} \subset V$, $u_n(x) = (-1)^n x(1 - x)$. The sequence is bounded ($\|u_n\|_\infty = 1/4$ for all n), but $\|u_{n+1} - u_n\|_\infty = 1/2$ for all n.

3. Let $V = C([0, 1])$ be equipped with the integral norm (A.18), and consider the sequence $\{u_n\}_{n=1}^{\infty} \subset V$, $u_n(x) = x^n$. It is $\|u_n\|_1 = 1/(n + 1)$ for all n, but the only function v that could be its limit is defined by $v(x) = 0$ for $x \in [0, 1)$ and $v(1) = 1$. However, this function does not lie in the space V. More about this situation will be said in Paragraph A.2.7.