1.1 SELECTED GENERAL PROPERTIES

Second-order PDEs (or PDE systems) encountered in physics usually are either elliptic, parabolic, or hyperbolic. Elliptic equations describe a special state of a physical system, which is characterized by the minimum of certain quantity (often energy). Parabolic problems in most cases describe the evolutionary process that leads to a steady state described by an elliptic equation. Hyperbolic equations describe the transport of some physical quantities or information, such as waves. Other types of second-order PDEs are said to be undetermined. In this introductory text we restrict ourselves to linear problems, since nonlinearities induce additional aspects whose understanding requires the knowledge of nonlinear functional analysis.

1.1.1 Classification and examples

Let $\mathcal{O}$ be an open connected set in $\mathbb{R}^n$. A sufficiently general form of a linear second-order PDE in n independent variables $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^T$ is

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial z_i}\left(a_{ij}\frac{\partial u}{\partial z_j}\right) + \sum_{i=1}^{n}\left(\frac{\partial}{\partial z_i}(b_i u) + c_i \frac{\partial u}{\partial z_i}\right) + a_0 u = f, \tag{1.1}$$

where $a_{ij} = a_{ij}(\boldsymbol{z}), b_i = b_i(\boldsymbol{z}), c_i = c_i(\boldsymbol{z}), a_0 = a_0(\boldsymbol{z})$ and $f = f(\boldsymbol{z})$. For all derivatives to exist in the classical sense, the solution and the coefficients have to satisfy the following regularity requirements: $u \in C^2(\mathcal{O}), a_{ij} \in C^1(\mathcal{O}), b_i \in C^1(\mathcal{O}), c_i \in C^1(\mathcal{O}), a_0 \in C(\mathcal{O})$, $f \in C(\mathcal{O})$. These regularity requirements will be reduced later when the PDE is formulated in the weak sense, and additional conditions will be imposed in order to ensure the existence and uniqueness of solution. If the functions a_{ij}, b_i, c_i, and a_0 are constants, the PDE is said to be with constant coefficients. Since the order of the partial derivatives can be switched for any twice continuously differentiable function u, it is possible to symmetrize the coefficients a_{ij} by defining

$$a_{ij}^{new} := (a_{ij}^{orig} + a_{ji}^{orig})/2$$

and adjusting the other coefficients accordingly so that the equation remains in the form (1.1). This is left to the reader as an exercise. Based on this observation, in the following we always will assume that the coefficient matrix $A(\boldsymbol{z}) = \{a_{ij}\}_{i,j=1}^n$ is symmetric.

Recall that a symmetric $n \times n$ matrix A is said to be positive definite if

$$\boldsymbol{v}^T A \boldsymbol{v} > 0 \qquad \text{for all } 0 \neq \boldsymbol{v} \in \mathbb{R}^n$$

and positive semidefinite if

$$\boldsymbol{v}^T A \boldsymbol{v} \geq 0 \qquad \text{for all } \boldsymbol{v} \in \mathbb{R}^n.$$

Analogously one defines negative definite and negative semidefinite matrices by turning the inequalities. Matrices which do not belong to any of these types are said to be indefinite.

Definition 1.1 (Elliptic, parabolic and hyperbolic equations) *Consider a second-order PDE of the form (1.1) with a symmetric coefficient matrix* $A(\boldsymbol{z}) = \{a_{ij}\}_{i,j=1}^n$.

1. *The equation is said to be* elliptic *at* $\boldsymbol{z} \in \mathcal{O}$ *if* $A(\boldsymbol{z})$ *is positive definite.*
2. *The equation is said to be* parabolic *at* $\boldsymbol{z} \in \mathcal{O}$ *if* $A(\boldsymbol{z})$ *is positive semidefinite, but not positive definite, and the rank of* $[A(\boldsymbol{z}), b(\boldsymbol{z}) + c(\boldsymbol{z})]$ *is equal to* n.